A note on a theorem of Armand Borel

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A NOTE ON A THEOREM OF ARMAND BOREL

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The theorem under discussion is the one which yields the cohomology of the classifying space of a Lie group. Let E be a canonical spectral algebra for cohomology over a field K with trivial E_∞ term, and let $B = \sum_p E_2^{p,0}$, $F = \sum_q E_2^{0,q}$ (the algebras corresponding to the cohomologies of base and fibre of a fibre space).

THEOREM 1. *If* $F = \Lambda(x_1, \ldots, x_m)$, *an exterior algebra on homogeneous elements of odd degree, then*

(a) homogeneous transgressive elements $y_1, \ldots, y_m$ *can be chosen such that*

[redacted]

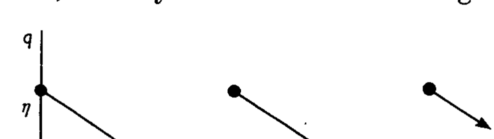
$$F = \Lambda(y_1, \ldots, y_m), \quad \textit{and} \quad \textit{degree } y_i = \textit{degree } x_i;$$

[redacted]

Definition of $E(s)$, an *elementary* spectral algebra over K of odd degree s. Let $F(s) = \Lambda(\eta)$, η of degree s, $B(s) = K[\zeta]$, ζ of degree $s+1$, and let $E(s)_2 = B(s) \otimes F(s)$. Therefore if K is not of characteristic 2, the algebra $E(s)_2$ is freely generated by ζ and η, [redacted]

$E(s)_2$ is generated by ζ and η, with the one relation $\eta^2 = 0$. Let $d_r = 0$, $r = 2, 3, \ldots, s$. [redacted]

$d_{s+1}(\zeta^k \otimes 1) = 0$ and $d_{s+1}(\zeta^k \otimes \eta) = \zeta^{k+1} \otimes 1$. Consequently, for $r > s+1$, $d_r = 0$ and $E(s)_r = E(s)_\infty =$ trivial, the only non-zero term in the bigrading being $E(s)_\infty^{0,0} = K$.



By construction $f:\bar{F} \cong F$. Therefore $f: \bar{B} \cong B$, qua graded groups, by the comparison theorem ((2), Dual corollary). But f is an algebra homomorphism, so that $f: \bar{B} \cong B$ is an algebra isomorphism. Consequently $B = K[z_1, \ldots, z_m]$.

Proof of Theorem 2. We recall that $F = \Delta(y_1, \ldots, y_m)$ means that the monomials [illegible] ($k = 1, 2, \ldots, m$) together with the unit element form an [illegible]

additive base for the vector space F over K. This is more general than an exterior algebra, since it may happen that $y_i^2 \neq 0$, as, for example, in the cohomology ring modulo 2 of the rotation group $R(3)$.

Since K is of characteristic 2 we may define elementary spectral algebras of even degree in exactly the same way as those of odd degree. For each i there is as before a spectral *sequence* homomorphism $f^i: E(s_i) \to E$, mapping η_i to y_i and ζ_i to z_i, only this time it is not strictly an *algebra* homomorphism, because although [illegible] $= 0$ we may have [illegible]