Extremal Graph Theory and Flag Algebra Exercises

1. Prove that a graph G is bipartite if and only if it contains no odd length cycles.

2. Show that if G is a graph with $\chi(G) = k$ then G has at least $\binom{k}{2}$ edges.

3. Prove that for any graph F the ,

$$\pi(F) = \lim_{n\to\infty} \frac{\text{ex}(n, F)}{\binom{n}{2}},$$

is well-defined.

4. Calculate the Turán density of the Petersen graph.

5. If F = $\{K , C , K_{,} \}$ what is π(F)?

6. Let $G = (V, E)$ be triangle-free with n vertices and suppose $\forall v \in V$ $d(v) > 2n/5$. Show that G is bipartite.

7. Prove that if $G = (V, E)$ is a graph with $n \geq 3$ vertices and $\lfloor n^2/4\rfloor + 1$ edges, then G contains at least $\lfloor n/2\rfloor$ triangles.

   Can you give an example to show that this is sharp?

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10. Let $S \subset \mathbb{R}^2$ with $|S| = n$ and suppose that $\|x - y\| \le 1$ for all $x, y \in S$. Show that

$$T = \left\{ (x, y) \in S \mid \|x - y\| > \frac{1}{\sqrt{2}} \right\}$$

satisfies $|T| \le \lfloor n^2/3 \rfloor$. (Hint: can you express this as a forbidden subgraph problem?)

Give an example to show that this bound is sharp.

11. Let

$$\Sigma^n = \{(x_1, x_2, \ldots, x_n) \mid x_1, \ldots, x_n \ge 0, \sum_{i}^{n} x_i = 1\}.$$

Given a graph $G = (V, E)$ with $V = [n]$ and $x \in \Sigma^n$, define $\lambda(G, x) = \sum_{ij \in E} x_i x_j$ and $\lambda(G) = \max_{x \in \Sigma^n} \lambda(G, x)$.

Suppose that $y = ($