Graph Theory Basics

A *graph* is a pair $G = (V\ E)$, consisting of a set of *vertices* V and a set of unordered pairs of vertices $E \subseteq V^{(2)}$ called *edges*.

If $v \in V(G)$ then the *neighbourhood* of v is $\Gamma(v) = \{w : vw \in E(G)\}$. The size of this neighbourhood is the *degree* of v denoted by $d(v)$.

Theorem 1 *If $G = (V\ E)$ is a graph then*

$$\sum_{v \in V} d(v) = 2|E|$$

Proof: Consider how many times each edge is counted in the LHS of this equation. □

Important examples of graphs include K_n, the *complete graph of order n*,

$$V(K_n) = [n] := \{1\ 2 \qquad n\} \qquad E(K_n) = [n]^{(2)} = \{ij : 1 \leq i \quad j \leq n\}$$

and the *cycle of length n*, C_n:

$$V(C_n) = [n] \qquad E(C_n) = \{i(i+1) : 1 \leq i \leq n-1\} \cup \{1n\}$$

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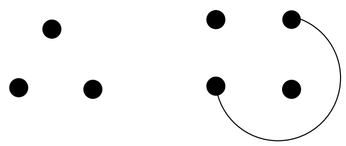

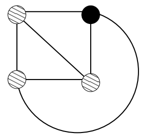

Figure 2: K_3 as a subgraph of K_4

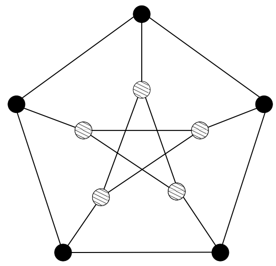

Figure 3: C_5 as a subgraph of the *Petersen graph*

A graph F is a *subgraph* of a graph G if there is an injective mapping $h : V(F) \to V(G)$ such that for all $uv \in E(F)$ we have $h(u)h(v) \in E(G)$. Moreover if h satisfies $uv \in E(F) \iff h(u)h(v) \in E(G)$ then we say that F is an *induced* subgraph.

If G has no subgraph that is isomorphic to F then we say G is F-*free*.

One of the main objectives of extremal graph theory is to calculate how many edges an F-free graph of order n may contain and so we define:

$$\text{ex}(n\ \ F) = \max\{|E(G)| : G \text{ is an } F\text{-free graph of order } n\}$$

Often this is too di□cult to compute so we instead aim to find the *Turán density*

$$(F) = \lim_{n\to\infty} \frac{\text{ex}(n\ \ F)}{\binom{n}{2}}$$

A k-*colouring* of a graph G is $c : V(G) \to [k]$ satisfying $uv \in E(G) \implies c(u) \neq c(v)$.

If a k-colouring of G exists we say that G is *k-partite*. A 2-partite graph is said to be *bipartite*. A special example of a bipartite graph is $K_{r,s}$, the complete bipartite graph with classes of size r and s:

$$V(K_{r,s}) = [r + s] \quad E(K_{r,s}) = \{ij : 1 \leq i \leq r \;\; r + 1 \leq j \leq r + s\}$$

The *chromatic number* of G is $(G) = \min\{k : G \text{ is } k\text{-partite}\}$. For example, $(K_t) = t$, while $(C_t) = 2$ if t is even and $(C_t) = 3$ if t is odd.

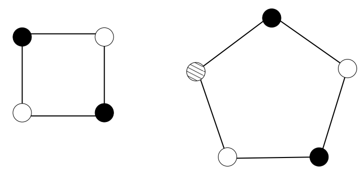

Figure 4: Colouring cycles

A *complete k-partite* graph is a k