# The many faces of polyhedra

Alexander P. Veselov (Loughborough, UK)

Polyhedra are very classical geometric objects. The 5 regular polyhedra known as platonic solids played a prominent role in Plato's philosophy and were the ultimate objects of study in Euclid's "Elements".

At the same time, even a rigorous mathematical definition of a polyhedron is not easy to give. In a way, polyhedra (and their multi-dimensional analogues – polytopes) are similar to integers: there are many natural questions one can ask, which turned out to be very hard. Nevertheless during the last few decades there were some striking findings in this area, which are not widely known.

I will talk about some of these remarkable results related to the following questions in this part of geometry, linking it to algebra, arithmetic, combinatorics and graph theory.

- Consider a polyhedron $P$ with vertices in integer lattice $\mathbb{Z}^3 \subset \mathbb{R}^3$. How many integer points are inside it? How is this number related to the volume of $P$? This is part of the Ehrhart theory of polytopes, which I will briefly sketch.

- Let $f_0, f_1, f_2$ be the number of vertices, edges and faces respectively of a convex polyhedron $P$. The integer vector $f(P) = (f_0, f_1, f_2)$ is called $f$-vector of $P$. Which integer vectors can be realised as $f$-vectors of some convex polyhedron/polytopes? Partial answer is given by the celebrated Dehn-Sommerville relations, which are far-going generalisations of the classical Euler relation
$$f_0 - f_1 + f_2 = 2.$$

If time permits I would also discuss the following question.

- For any polyhedron $P$ one can define its skeleton - graph $\Gamma(P)$ formed by its vertices and edges. Can one describe all the graphs $\Gamma$, which can be realised in this way? What is the "best shape" of a polyhedron $P$ with given $\Gamma = \Gamma(P)$?

No specific knowledge beyond linear algebra will be required, I will introduce and explain all the necessary material.

### Recommended literature.

[1] M. Beck, S. Robins *Computing the continuous discretely. Integer-point enumeration in polyhedra.* Springer, 2007.

[2] G.M. Ziegler *Lectures on polytopes.* Springer, 1995.